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remainder 1 from 7. The final remainder 6 represents the sixth letter of the alphabet which is *F*; and therefore *F* is the Dominical or Sunday letter for the last ten months of the year 1776; for 1776 was a leap year, and Uncle Zadock's rule always gives you the Sunday letter for the last ten months of leap years. *F* being the Sunday letter for the last ten months of the year, and the letter for July always being *G*, it follows that July 1, 1776, fell on Monday and July 4 on Thursday.

We may note that within a given century the first three steps in the process will always yield the same result. Thus in the nineteenth century there will always be two and in the twentieth century nothing to be added to the odd numbers and the fourth part of them. In applying this rule to any date within the present century, we may simply ignore the centurial number entirely and proceed with the fourth step in the rule.

II. CONSTRUCTION OF THE REGULAR UNDECAGON BY A SEXTIC CURVE.

By CYRUS B. HALDEMAN, Ross, Butler County, Ohio.

Consider the sextic

$$x(a^2+3ay^2+y^4)\sqrt{-(b^2+a^{11})} = a^5y^6-by^5+6a^6y^4-5aby^3+9a^7y^2-5a^2by+2a^8, \tag{1}$$

which will be real when b^2+a^{11} is negative, and the circle

$$x^2+y^2=-4a, \tag{2}$$

which will be real when a is negative.

Eliminate x from (1) by means of (2). After expansion, reduction, and separation into factors, the result may be written in the form

$$(y^{11}+11ay^9+44a^2y^7+77a^3y^5+55a^4y^3+11a^5y+2b)(a^5y-2b)=0.$$

By the transformation $y=-2s\sqrt{-a}$, the first of these factors placed equal to zero gives the equation

$$11s-220s^3+1232s^5-2816s^7+2816s^9-1024s^{11}=\frac{b}{a^5\sqrt{-a}};$$

and because

$$11\sin A-220\sin^3 A+1232\sin^5 A-2816\sin^7 A + 2816\sin^9 A-1024\sin^{11} A=\sin 11A,$$

we may take

$$s=\sin A, \qquad \text{where} \qquad \sin 11A=\frac{b}{a^5\sqrt{-a}},$$

and get

$$y=-2\sqrt{-a}\,\sin\frac{1}{11}\sin^{-1}\frac{b}{a^5\sqrt{-a}},$$

$$y = -2\sqrt{-a}\sin\frac{1}{11}\left(2\pi + \sin^{-1}\frac{b}{a^5\sqrt{-a}}\right),$$

. . . .

$$y = 2\sqrt{-a}\sin\frac{1}{11}\left(2\pi - \sin^{-1}\frac{b}{a^5\sqrt{-a}}\right).$$

This solution indicates that the eleven intersections whose ordinates are the eleven real roots of the above-mentioned factor are the vertices of the regular undecagon required.

The above factor of the eleventh degree may be resolved algebraically by the transformation $y = x - a/x$. The result is

$$y = \sqrt[11]{-b + \sqrt{b^2 + a^{11}}} + \sqrt[11]{-b - \sqrt{b^2 + a^{11}}}.$$

Remarks by the Editor.

The interest of this problem is not in the mere determination of a sextic through the vertices. It is easy to find a set of six straight lines through the eleven points, and to rotate the set about the origin to obtain the various positions of the undecagon. The problem consists rather in the determination of a curve of the special type $xY_4 = Y_6$, where Y_i denotes a polynomial in y of degree i. Now Y_6/Y_4 is a rational function of y involving eleven constants, so that this may be looked upon as a case of Cauchy's general problem of interpolation by rational functions.[1] The coefficients being determined as symmetric functions of the coördinates of the vertices, we should, after simplification, have the equation (1).

This would, however, be a tedious procedure, and the problem invites a more special treatment. Mr. Haldeman does not give the steps by which he was led to his sextic equation, though it would seem to be through some use of the formula giving sin 11θ in terms of sin θ. It may also be derived from the formulas for 5θ and 6θ, as the following remarks show.

Let a regular polygon of $4n + 3$ sides be inscribed in the circle $x^2 + y^2 = c^2$, and let one of the vertices have vectorial angle α. Then the vectorial angle θ of any vertex satisfies the equation

$$\cos\{(2n+1)\theta - (4n+3)\alpha\} = \cos(2n+2)\theta,$$

or

$$\cos(4n+3)\alpha\cos(2n+1)\theta = \cos(2n+2)\theta - \sin(4n+3)\alpha\sin(2n+1)\theta.$$

Now $\cos(2n+1)\theta/\cos\theta$, $\cos(2n+2)\theta$, and $\sin(2n+1)\theta$ may be expressed[2] as polynomials in $\sin\theta$ of degrees $2n$, $2n+2$, and $2n+1$. Hence, on writing x/c for $\cos\theta$ and y/c for $\sin\theta$, we obtain an equation of the type

$$xY_{2n} = Y_{2n+2}, \qquad (A)$$

which is equivalent to (1) when $n = 2$.

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[1] G. Boole, *A Treatise on the Calculus of Finite Differences*, London, 1872, p. 43.

[2] E. W. Hobson, *Plane Trigonometry*, edition 2, Cambridge, 1897, pp. 104–106.

Equation (A) is the general equation of a curve of degree $2n + 2$ having at infinity a point of multiplicity $2n + 1$, whose $2n + 1$ tangents are the line at infinity and $2n$ lines parallel to the x-axis. Such a curve is uniquely determined when c and α are given. The positions of the asymptotes $Y_{2n} = 0$ are independent of α.

It is not difficult to trace the changes in the form of the curve as α varies. It makes $4n + 4$ real intersections with the circle, the extra point having a vectorial angle $-(4n + 3)\alpha$. This extra point always crosses an asymptote at the same time as a vertex crosses the same asymptote on the opposite side of the circle, and the curve then becomes composite, the asymptote being a part of the curve. When no vertex is on an asymptote, the strips into which the asymptotes divide the plane contain two vertices each, one to the right and one to the left, except that the strip containing the extra intersection contains three vertices. This strip also contains (if $\alpha \neq \pm \pi/2$) the only point on the curve where the tangent is parallel to the y-axis. When $\alpha = \pm \pi/2$ (that is, when in equation (1) $b^2 = -a^{11}$), the curve becomes six straight lines.

To obtain similar results for a regular polygon of $4n + 1$ sides (as in Mr. Haldeman's former article on the pentagon) we may start from the equation

$$\sin \{2n\theta - (4n + 1)\alpha\} = - \sin (2n + 1)\theta,$$

and proceed in very much the same way.

An equally simple type of curve (for a polygon of m sides) is $xY_{m-1} = Y_{m-1}$, that is a curve of degree m with $m - 1$ asymptotes parallel to the x-axis and one parallel to the y-axis. To get this, we use, if $m = 4n + 3$, the equation

$$\sin \{(2n + 1)\theta - (4n + 3)\alpha\} = - \sin (2n + 2)\theta;$$

and, if $m = 4n + 1$, the equation

$$\cos \{2n\theta - (4n + 1)\alpha\} = \cos (2n + 1)\theta.$$

III. A GENERALIZATION OF THE PYTHAGOREAN THEOREM.

By J. ROSENBAUM, Bloomfield, Conn.

There is a given angle XOY, and a given number n, and we look for pairs of points, P and Q on OX and OY respectively, such that

$$\overline{PQ}^n = \overline{PO}^n + \overline{OQ}^n.$$

When the given angle is a straight angle or a right angle, such a pair of points does not exist unless $n = 1$ or $n = 2$, respectively, and for these values of n any pair of points will satisfy the condition. Conversely, if $n = 1$ or $n = 2$, there is no solution unless the angle is a straight angle or a right angle respectively, and for these angles any pair of points will do. Furthermore, for n such that $0 \leqq n < 1$ there is no solution, because then the condition would require either that PQ be greater than $PO + OQ$, or that 1 be equal to $1 + 1$.